Supplemental Notes

EE503 Week 04 Dr. Franzke

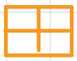
HW # 4

1. Leon-Garcia 3.53 – 3.54

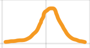
   3.57 – 3.59

Topics

1. BEG CUP (acronym for 6 main pdfs)

B: Binomial 

E: Exponential γ, χ^2

G: Gaussian  thin-tails

C: Cauchy SαS

U: Uniform Beta, Dirichlet.

P: Poisson Counting structure

Poisson pdf

BEG CUP

issue spot: counting structure

$X \sim P(\lambda)$ for parameter ("mean") $\lambda > 0$

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

Thrm (Poisson Law) $b \xrightarrow{d} P$ if $n >> 1$ and $p << 1$ and $\lambda = np$.

Pf: $\lim_{n\to\infty} b(n,k,p) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$.

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k} \quad \text{if } \lambda = np.$$

$$= \frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n!}{(n-k)!\,n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n\cdot(n-1)\cdots(n-k+1)}{n \quad n \quad \cdots \quad n}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!}\lim_{n\to\infty} 1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k+1}{n}\right)\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!}\underbrace{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)}_{=1}\cdots\underbrace{\lim_{n\to\infty}\left(1-\frac{k+1}{n}\right)}_{=1}\cdot\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n}_{e^{-\lambda}}\cdot\underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k}}_{=1}$$

$$= \frac{\lambda^k}{k!}e^{-\lambda}$$

$$= P(\lambda)$$

QED.

Ex: Alpha-decay of Radium to Radon

$$^{226}_{88}Ra \longrightarrow {}^{222}_{86}Rn + {}^{4}_{2}He^{2+}$$

α-particle

half-life of ^{226}Ra isotope: 1600 years

Radium: 226 grams / mole

$\therefore$ 1 gram $\approx 10^{22}$ atoms $(= n)$

Binary event: atom "decays" spontaneously or not.

"Vast inter-atomic distance" $\longrightarrow$ independence

$\lambda =$ α-emissions per-gram per-second

$\approx 10^{10}$ decays

$T = 1$ second

$$\therefore P(1 \text{ α-decay}) = b(10^{25}, 1) \approx \frac{10^{10}}{10^{22}} = 10^{-12}$$

$$= \frac{1}{\text{trillion}}$$

$$\therefore P(X(t) = k) \approx \frac{e^{-\lambda}\lambda^k}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

Ex: Basketball long shots $X \sim b(70, 0.02)$

assume independent shots

$\therefore \begin{cases} n = 70 \gg 1 \\ p = 0.02 \ll 1 \end{cases}$

$\therefore \lambda = n \cdot p = 70 \cdot (0.02) = 1.4$

$\therefore$ Exact binomial probability

$$P(3 \text{ baskets in } 70 \text{ trials}) = \binom{70}{3}(0.02)^3(0.98)^{67}$$

$$= 0.1131$$

$\therefore$ Poisson approximation with $\lambda = np = 1.4$

$$P(X=3) = \frac{(1.4)^3 e^{-1.4}}{3!} = 0.1128$$

"moment generating function"

<u>Later</u> $\quad b(n,k,p) \longleftrightarrow \mathcal{M}_B(s) = (1-p+pe^s)^n$

$$\left(= E[e^{sX}] \right)$$

$$P(\lambda) \longleftrightarrow \mathcal{M}_P(s) = e^{\lambda(e^s-1)}$$

$\therefore$ For $n \cdot p = \lambda$ constant.

$$\lim_{n\to\infty} \mathcal{M}_B(s) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n}e^s\right)^n \qquad \text{since } p = \frac{\lambda}{n}$$

$$= \lim_{n\to\infty} \left(1 - \frac{\lambda(e^s-1)}{n}\right)^n$$

$$= e^{\lambda(e^s-1)}$$

$$= \mathcal{M}_P(s)$$

$\therefore$ Again: $b(n,k,p) \xrightarrow{d} P(\lambda)$ if $\lambda = n \cdot p$ $\quad$ (and Levy's Theorem week 11)

Accuracy of Poisson Approximation

Okay: $\quad n \geq 20$ and $p \leq 0.05$

Excellent: $\quad n \geq 100$ and $p \leq 0.1$

BEG CUP

Sampling

Outcomes	w/ replacement	w/o replacement
2	Binomial negative binomial geometric	Hypergeometric "until"
≥ 2	Multinomial	Multivariate Hypergeometric

Hypergeo → Binomial

Multi Hypergeo → Multinomial

$$\frac{\binom{N_1}{n_1}\binom{N_2}{n_2}\cdots\binom{N_k}{n_k}}{\binom{N}{n}}$$

$n = n_1 + \cdots + n_k$

$N = N_1 + \cdots + N_k$

Issue-Spotting Sequence

Q1: Outcomes? (2 or more?)

Q2: Sampling? (w/ or w/o replacement)

Q3: Until structure?

"UNTIL" ⟹ negative binomial

(k=1 outcomes, geometric)

UNTIL

Discrete → Negative Binomial (k=1: Geometric)

Continuous → γ (gamma)

(k=1: Exponential

k>1: Erlang (k ∈ $\mathbb{Z}^+$))

Fact: $\binom{N}{n} = \sum_{k=0}^{n} \binom{N_1}{k}\binom{N_2}{n-k}$ if $N = N_1 + N_2$

and $\binom{k}{j} = 0$ if $j > k$

Thrm Hypergeometric $\xrightarrow{d}$ Binomial if $N \to \infty$ and $p = \frac{N_1}{N}$

Prf: Say $X \sim Hyp(n, m, k, N)$. Fixed k & n

$$\therefore \lim_{N\to\infty} P(X=k) = \lim_{N\to\infty} \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$$= \lim_{N\to\infty} \frac{m!}{k!\,(m-k)!} \cdot \frac{(N-n)!\,n!}{N!} \cdot \frac{(N-m)!}{(n-k)!\,(N-n-m+k)!}$$

$$= \lim_{N\to\infty} \frac{\frac{n!}{(n-k)!\,k!} \cdot \frac{m!\,(N-m)!}{(N-m-n+k)!\,(m-k)!}}{\frac{N!}{(N-n)!}}$$

$$= \lim_{N\to\infty} \binom{n}{k} \frac{\frac{m!}{(m-k)!} \cdot \frac{(N-m)!}{(N-m-n+k)!}}{\frac{N!}{(N-n)!}}$$

$$= \binom{n}{k} \lim_{N\to\infty} \frac{\prod_{j=1}^{k}(m-k+j) \prod_{j=1}^{n-k}(N-m-n+k+j)}{\prod_{j=1}^{n}(N-n+j)} \cdot \frac{N^n}{N^n}$$

$\binom{n}{k}$ fixed

$$= \binom{n}{k} \cdot \lim_{N\to\infty} \frac{\prod_{j=1}^{k}\Big(\overbrace{\tfrac{m}{N}}^{p} - \cancel{\tfrac{k}{N}} + \cancel{\tfrac{j}{N}}\Big) \prod_{j=1}^{n-k}\Big(\overbrace{1 - \tfrac{m}{N}}^{1-p} - \cancel{\tfrac{n}{N}} + \cancel{\tfrac{k}{N}} + \cancel{\tfrac{j}{N}}\Big)}{\prod_{j=1}^{n}\Big(1 - \cancel{\tfrac{n}{N}} + \cancel{\tfrac{j}{N}}\Big)}$$

$$= \binom{n}{k} \cdot \left(\frac{m}{N}\right)^k \left(1 - \frac{m}{N}\right)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

$$= b(n, k, p)$$

QED

$X \sim NB$: Negative Binomial

$$P(X = x \text{ trials until } k \text{ successes}) = \binom{x-1}{k-1} p^k q^{x-k}$$

$\therefore$ $k = 1$ success : $X \sim$ Geometric

$$P(X = x \text{ until first success}) = \binom{x-1}{0} p^1 q^{x-1}$$

$$= p \cdot q^{x-1}$$

Thrm: $$\sum_{k=0}^{n} a^k = \begin{cases} n+1 & \text{if } a = 1 \\ \dfrac{1-a^{n+1}}{1-a} & \text{if } a \neq 1. \end{cases}$$

Prf: $$S \stackrel{\Delta}{=} a^0 + a^1 + \cdots + a^n = \sum_{k=0}^{n} a^k$$

$$\therefore aS = a + a^2 + \cdots + a^{n+1}$$

$$\therefore S(1-a) = S - a \cdot S$$

$$= (1 + a + \cdots + a^n) - (a + a^2 + \cdots + a^{n+1})$$

$$= 1 - a^{n+1}$$

$$\therefore S = \frac{1-a^{n+1}}{1-a} \quad \text{since } a \neq 1$$

QED

Corr 1: $$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \quad \text{if } |a| < 1$$

Corr 2: $$\sum_{k=1}^{\infty} a^k = \frac{a}{1-a} \quad \text{if } |a| < 1$$

Corr 3: $$\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a} \quad \text{if } |a| < 1$$

Thrm: $P(X > k) = q^k$ if $X \sim G(p)$

$\therefore P(X \le k) = 1 - q^k$

Prf: $P(X > k) = P(\{X = k+1\} \underset{\text{"OR"}}{\cup} \{X = k+2\} \underset{\text{"OR"}}{\cup} \cdots)$

$$\overset{\text{CAT}}{=} \sum_{x=k+1}^{\infty} p(x)$$

$$\overset{X\sim G(p)}{=} \sum_{x=k+1}^{\infty} p \cdot q^{x-1} \overset{\substack{y = x-1 \\ \therefore x = y+1}}{=} p \cdot \sum_{y=k}^{\infty} q^y$$

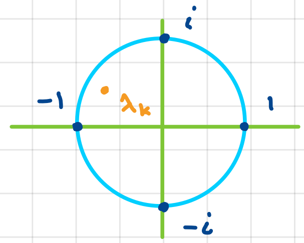
$$\overset{\text{Corr3}}{=} p \cdot \frac{q^k}{1-q} = \frac{p}{p} \cdot q^k = q^k$$

QED.

(Optional) Matrix Version: $\underset{n\times n}{A} \in \mathbb{C}^{n \times n}$

Thrm: $\lim_{n \to \infty} A^n = \vec{0}$ iff $|\lambda_k| < 1$ for all eigenvalues λ_k of A.

($Ax = \lambda x$ if $x \neq 0$)



If all λ_k in unit circle

$$\sum_{k=0}^{n} A^k = (I - A)^{-1}(I - A^{n+1})$$

where $I_{n\times n} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\therefore \sum_{k=0}^{\infty} A^k = (I - A)^{-1}$$

$\therefore$ Generalizes $\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}$ when $|a| < 1$

Thm: (Stirling's Approximation) $n! \sim \sqrt{2\pi n}\,\frac{n^n}{e^n}$

Prf: $n! = \Gamma(n+1)$

$= \int_0^\infty x^n e^{-x}\,dx$ put $u = \frac{x-n}{\sqrt{n}}$

$\therefore x = n + \sqrt{n}\,u$ and $dx = \sqrt{n}\,du$

$x = 0 \longrightarrow u = -\frac{n}{\sqrt{n}} = -\sqrt{n}$

$x \uparrow \infty \longrightarrow u \uparrow \infty$

$$= \int_{u=-\sqrt{n}}^{u=\infty} (n+\sqrt{n}\,u)^n e^{-n-\sqrt{n}\,u} \sqrt{n}\,du$$

$$= \sqrt{n}\,e^{-n} \int_{-\sqrt{n}}^{\infty} (n+\sqrt{n}\,u)^n e^{-\sqrt{n}\,u}\,du$$

$$= \sqrt{n} \cdot \frac{n^n}{e^n} \int_{-\sqrt{n}}^{\infty} \left(1+\frac{1}{\sqrt{n}}u\right)^n e^{-\sqrt{n}\,u}\,du$$

But: $\lim_{n\to\infty} \int_{-\sqrt{n}}^{\infty} \left(1+\frac{1}{\sqrt{n}}u\right)^n e^{-\sqrt{n}\,u}\,du = \sqrt{2\pi}$

(advanced)

$\therefore$ by Lebegue's Dominated Convergence Theorem

$$\therefore \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,\frac{n^n}{e^n}} = 1$$

$$\therefore n! \sim \sqrt{2\pi n}\,\frac{n^n}{e^n}$$

QED

Ex: Flip a coin n-times with $p = P(H)$. Suppose n is even. What is the probability of getting $\frac{n}{2}$ heads in n flips

Answer: $b(n, \frac{n}{2}, p) \approx \sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{n}}\,(4pq)^{n/2}$

$$\therefore p = q = \tfrac{1}{2} \Longrightarrow b(n,k,p) \approx \sqrt{\tfrac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$$

$\therefore$ Slow (square-root) decay.

Prf: $b(n,k,p) = \binom{n}{n/2} p^{n/2} q^{n-n/2}$ since n · even

$$= \frac{n!}{(n-n/2)!\,(n/2)!} (pq)^{n/2}$$

$$= \frac{n!}{((n/2)!)^2} (pq)^{n/2}$$

$$\approx \frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}\,(p\cdot q)^{n/2}}{\left(\sqrt{2\pi \frac{n}{2}}\,\left(\frac{n}{2}\right)^{n/2} e^{-n/2}\right)^2}$$ by Stirling's Approximation

$$= \frac{\sqrt{2\pi n}\,\frac{n^n}{e^n}\,(pq)^{n/2}}{\pi \cdot n \cdot \frac{n^n}{2^n} e^{-n}}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}\, 2^n\, (pq)^{n/2}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}} (4 \cdot p \cdot q)$$

$\therefore\ p = q = \frac{1}{2} \longrightarrow (4pq)^{n/2} = 1^{n/2} = 1$

$\therefore\ b(n,k,p) \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}$

QED.

Ex: (computer)

$b(10,5,\frac{1}{2}) = 0.24609$ dbinom(5,10,0.5)

faster decay if $p \neq q$

Stirling: $b(10,5,\frac{1}{2}) \approx 0.25231$ sqrt(2/pi/10)

n	$b(n,\frac{n}{2},\frac{1}{2})$	Stirling's Approx.
100	0.03958	0.03978
1000	0.02522	0.02523
10,000	0.00798	0.00798
100,000	0.00252	0.00252